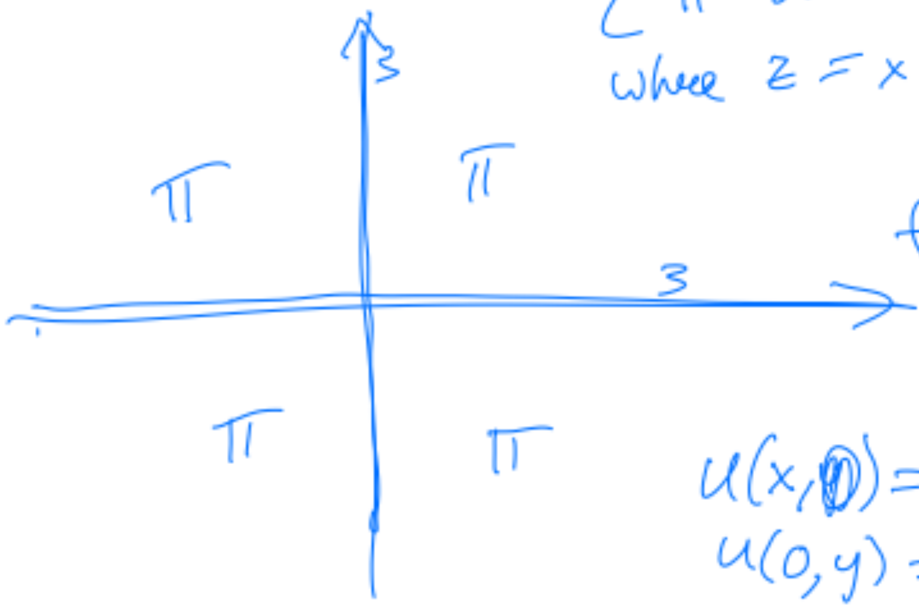
Example where the Cauchy Riemann equations hold at $z=0$ but the function is not continuous at $z=0$:

let $f(z) = \begin{cases} 3 & \text{if } y=0 \text{ or } x=0 \\ \pi & \text{otherwise,} \end{cases}$

where $z = x+iy$ for $x,y \in \mathbb{R}$.



$f(z) = u(x,y) + i\,v(x,y)$

$u(x,0) = 3$

$u(0,y) = 3$

$v(x,0) = 0$

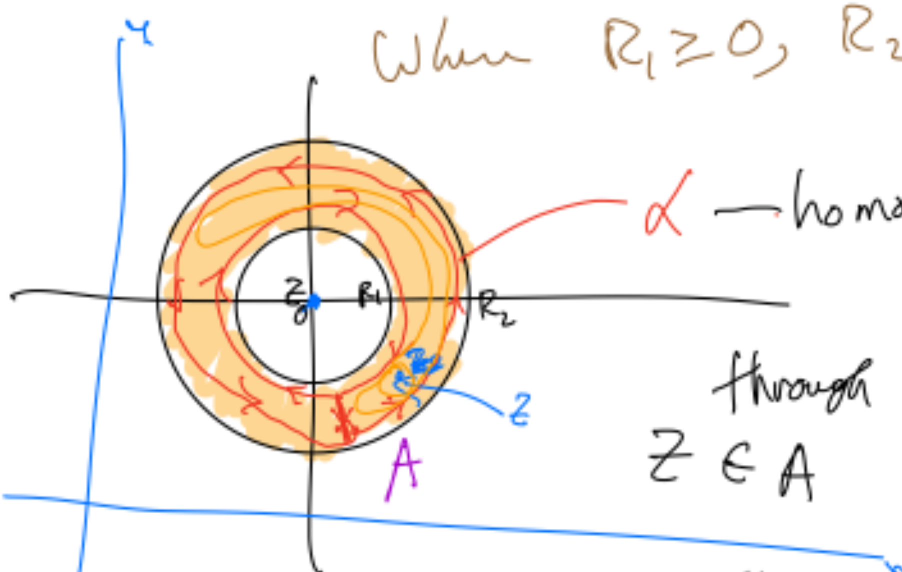
$v(0,y) = 0.$

$u_x(0,0) = 0 = v_y(0,0) = 0$

$u_y(0,0) = 0 = -v_x(0,0) = 0.$

From last time:

Suppose f is a holomorphic function that is defined on an annulus $A = \{z \in \mathbb{C} : R_1 < |z - z_0| < R\}$ when $R_1 \geq 0$, $R_2 > R_1 \geq 0$.



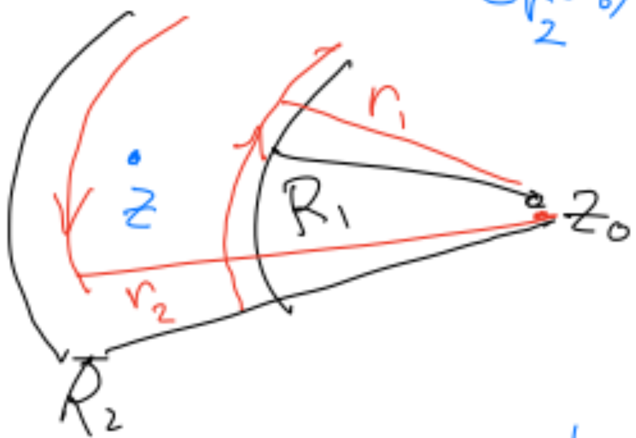
α — homotopic to $\partial B(z, \varepsilon) = \tilde{\alpha}$ through domain A.

$z \in A$

$$f(z) = \frac{1}{2\pi i} \int_\alpha \frac{f(w)}{w - z}\, dw$$

(by CIF)

Now, $\displaystyle\int_{\alpha} = \int_{C_{r_2}(z_0)} + -\int_{C_{r_1}(z_0)}$



Where $R_1 < r_1 < |z - z_0| < r_2 < R_2$

$$\therefore\ f(z) = \frac{1}{2\pi i}\int_{\alpha} \frac{f(\omega)}{\omega - z}\,d\omega =$$

$$\frac{1}{2\pi i}\int_{\substack{C_{r_2}(z_0)\\ \text{(outer)}}} \frac{f(\omega)}{\omega - z}\,d\omega - \frac{1}{2\pi i}\int_{\substack{C_{r_1}(z_0)\\ \text{(inner)}}} \frac{f(\omega)}{\omega - z}\,d\omega$$

by deformation Thm.

We will now rewrite the integrand in terms of $(z - z_0)$.

$$\frac{f(w)}{w-z} = \frac{f(w)}{(w-z_0)-(z-z_0)}$$

We will now rewrite so it looks like a geometric series.

Two options

$$\frac{f(w)}{w-z} = \frac{f(w)}{(w-z_0)} \cdot \frac{1}{1-\left(\frac{z-z_0}{w-z_0}\right)}$$

gives geometric series if $|z-z_0| < \underbrace{|w-z_0|}_{\text{radius}}$

works if $w \in C_{r_2}(z)$

OR

$$\frac{f(w)}{w-z} = \frac{-f(w)}{(z-z_0)} \cdot \frac{1}{1-\left(\frac{w-z_0}{z-z_0}\right)}$$

gives geometric series if $\underbrace{|w-z_0|}_{\text{radius}} < |z-z_0|$

Works for $w \in C_{r_1}(z_0)$

$$\therefore f(z) = \frac{1}{2\pi i}\int_{C_{r_2}(z_0)} \frac{f(w)}{(w-z_0)} \cdot \frac{1}{1-\left(\frac{z-z_0}{w-z_0}\right)}\,dw$$

$$+ \frac{1}{2\pi i}\int_{C_{r_1}(z_0)} \frac{f(w)}{(z-z_0)} \cdot \frac{1}{1-\left(\frac{w-z_0}{z-z_0}\right)}\,dw$$

Now we expand (substitute the convergent geometric series's)

$$\Rightarrow f(z) = \frac{1}{2\pi i}\int_{C_{r_2}(z_0)} \frac{f(w)}{(w-z_0)} \sum_{k\geq 0} \left(\frac{z-z_0}{w-z_0}\right)^k dw$$

$$+ \frac{1}{2\pi i}\int_{C_{r_1}(z_0)} \frac{f(w)}{(z-z_0)} \sum_{m\geq 0} \left(\frac{w-z_0}{z-z_0}\right)^m dw$$

Since we have absolute & uniform convergence of the geometric series at these radii, we can pull out the sum:

$$f(z) = \sum_{k\geq 0}\left(\frac{1}{2\pi i}\int_{C_{r_2}(z_0)} \frac{f(w)}{(w-z_0)^{k+1}}\,dw\right)(z-z_0)^k$$

$$+ \sum_{m\geq 0}\left(\frac{1}{2\pi i}\int_{C_{r_1}(z_0)} f(w)\cdot(w-z_0)^m\,dw\right)(z-z_0)^{-m-1}$$

$$\therefore\ f(z) = \sum_{k\geq 0} c_k (z-z_0)^k + \sum_{m\geq 0} b_m (z-z_0)^{-m-1},$$

where $c_k = \frac{1}{2\pi i}\int_{C_{r_2}(z_0)} \frac{f(w)\,dw}{(w-z_0)^{k+1}}$ $\quad r_2 > |z-z_0|$

and $b_m = \frac{1}{2\pi i} \int_{C_{r_1}(z_0)} f(\omega)(\omega - z_0)^m \, d\omega$.

$r_1 < |z - z_0|$

In summary,

$$f(z) = \sum_{k \in \mathbb{Z}} a_k (z - z_0)^k,$$

where $a_k = \frac{1}{2\pi i} \int_{C_r(z_0)} \frac{f(\omega)}{(\omega - z_0)^{k+1}} \, d\omega$,

where $R_2 > r > |z - z_0|$ for $k \geq 0$,

$R_1 < r < |z - z_0|$ for $k < 0$.

Laurent series.

Note: If f is actually holomorphic on $B_r(z_0)$ where $r > |z - z_0|$.
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ $\| R$.

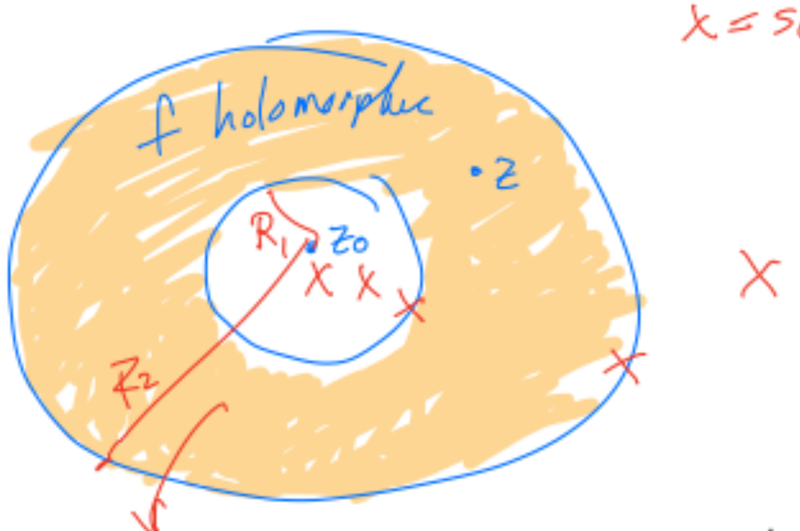
(Holomorphic on $\overline{B(z_0, R)}$), then $a_k = 0$ for $k \leq -1$

$$\frac{1}{2\pi i}\int_{C_r(z_0)} \frac{f(w)}{(w-z_0)^{k+1}}\,dw = 0 \quad \text{since} \quad f(w)(w-z_0)^{(\geq 0)} \text{ is holomorphic.}$$

In this case
So we get the usual Taylor series.

Important: The Laurent series converges to the holomorphic function on that annulus. You can choose the annulus of convergence to be maximal

x = singularity of f



$A(z_0, R_1, R_2)$ ← The Laurent series converges exactly on this maximal annulus to $f(z)$, where the function f can not be extended to a holomorphic fcn on any bigger annulus. (The Laurent series diverges for $|z-z_0| < R_1$ or $|z-z_0| > R_2$)

<u>Fact</u>: Given an annulus around z_0 where a holomorphic function is defined, The Laurent series for that annulus is uniquely determined. (Because the coefficients are uniquely determined by the function & annulus.)

<u>Simple examples.</u>

① Plug $\frac{1}{z}$ or $\frac{1}{z-z_0}$ into any holomorphic fcn.

eg. $e^z = \sum_{k \geq 0} \frac{z^k}{k!}$ ← converges for all $z \in \mathbb{C}$.

$\Rightarrow e^{1/z} = \sum_{k \geq 0} \frac{\left(\frac{1}{z}\right)^k}{k!}$ ← converges if $\frac{1}{z} \in \mathbb{C}$. ie $\forall z \in \mathbb{C} \setminus \{0\}$.

$= \sum_{k \geq 0} \frac{z^{-k}}{k!}$

Annulus of convergence $R_1 = 0 < |z| < \infty = R_2$

$m = -k$ from above

$$e^{1/z} = \sum_{m \le 0} \frac{z^{m}}{(-m)!} = 1 + \frac{z^{-1}}{1!} + \frac{z^{-2}}{2!} + \frac{z^{-3}}{3!} + \dots$$

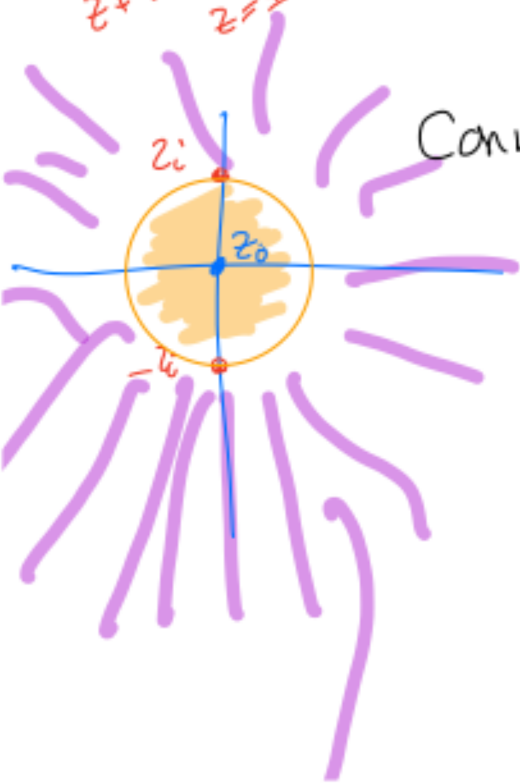
$\boxed{\text{Ex}}$ Consider $f(z) = \dfrac{1}{z^2+4}$. Find all possible Laurent series & annuli centered at $z=0$.

(1) $$g(z) = \frac{1}{z^2+4} = \frac{1}{4}\,\frac{1}{\left(1-\left(\frac{-z^2}{4}\right)\right)} = \frac{1}{4}\sum_{k\ge 0}\left(\frac{-z^2}{4}\right)^k$$

$$= \sum_{k \ge 0} \frac{(-1)^k}{4^{k+1}} z^{2k}.$$

Converges for $\left|\frac{-z^2}{4}\right| < 1 \iff |z| < 2$

Singular when $z^2+4=0$, i.e $z = \pm 2i$



Converges on the annulus

$$0 \le |z| < 2$$

② $2<|z|<\infty$ (no other singularities).

$$g(z)=\frac{1}{z^2+4}=\frac{1}{z^2}\left(\frac{1}{1-\left(\frac{-4}{z^2}\right)}\right)$$

$$\left|\frac{-4}{z^2}\right|<1 \iff |z|>2. \checkmark$$

$$=\frac{1}{z^2}\sum_{k\geq 0}\left(\frac{-4}{z^2}\right)^k=\boxed{\sum_{k\geq 0}(-4)^k\cdot z^{-2k-2}}$$

We can apply this for real #s:

$$\frac{1}{4+x^2}=\sum_{k\geq 0}\frac{(-1)^k}{4^{k+1}}x^{2k}$$

$$=\frac{1}{4}-\frac{x^2}{4^2}+\frac{x^4}{4^3}-\frac{x^6}{4^4}+\cdots$$

valid $|x|<2$

$$\frac{1}{4+(.01)^2}=\frac{1}{4}-\frac{(.01)^2}{4^2}+\frac{(.01)^4}{4^3}-\frac{(.01)^6}{4^4}+\cdots$$

$$=\frac{1}{4}-\frac{.0001}{4^2}+\frac{.00000001}{4^3}-\cdots$$

What if we want to write $\frac{1}{4+6^2}$ as a series?

$$\frac{1}{4+x^2} = \sum_{k\geq 0} (-4)^k x^{-2k-2}$$

$|x|=|6|>2$

$$g(6) = \frac{1}{x^2} - \frac{4}{x^4} + \frac{4^2}{x^6} - \frac{4^3}{x^8} + \cdots$$

$$\frac{1}{40} = \frac{1}{6^2} - \frac{4}{6^4} + \frac{4^2}{6^6} - \frac{4^3}{6^8} + \cdots$$